

# A PRIORI ESTIMATES AND APPLICATION TO THE SYMMETRY OF SOLUTIONS FOR CRITICAL $p$ -LAPLACE EQUATIONS

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**ABSTRACT.** We establish pointwise a priori estimates for solutions in  $D^{1,p}(\mathbb{R}^n)$  of equations of type  $-\Delta_p u = f(x, u)$ , where  $p \in (1, n)$ ,  $\Delta_p := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator, and  $f$  is a Caratheodory function with critical Sobolev growth. In the case of positive solutions, our estimates allow us to extend previous radial symmetry results. In particular, by combining our results and a result of Damascelli–Ramaswamy [5], we are able to extend a recent result of Damascelli–Merchán–Montoro–Sciunzi [6] on the symmetry of positive solutions in  $D^{1,p}(\mathbb{R}^n)$  of the equation  $-\Delta_p u = u^{p^*-1}$ , where  $p^* := np/(n-p)$ .

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we are interested in problems of the type

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \mathbb{R}^n, \\ u \in D^{1,p}(\mathbb{R}^n), \end{cases} \quad (1.1)$$

where  $p \in (1, n)$ ,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $D^{1,p}(\mathbb{R}^n)$  is the completion of  $C_c^\infty(\mathbb{R}^n)$  with respect to the norm  $\|u\|_{D^{1,p}(\mathbb{R}^n)} := (\int_{\mathbb{R}^n} |\nabla u|^p dx)^{1/p}$ , and  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  is a Caratheodory function such that

$$|f(x, s)| \leq \Lambda |s|^{p^*-1} \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n, \quad (1.2)$$

for some real number  $\Lambda > 0$ , with  $p^* := np/(n-p)$ .

Our main result is as follows.

**Theorem 1.1.** *Let  $p \in (1, n)$ ,  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that (1.2) holds true and  $u$  be a solution of (1.1). Then there exists a constant  $C_0 = C_0(n, p, \Lambda, u)$  such that*

$$|u(x)| \leq C_0 (1 + |x|^{\frac{n-p}{p-1}})^{-1} \quad \text{and} \quad |\nabla u(x)| \leq C_0 (1 + |x|^{\frac{n-1}{p-1}})^{-1} \quad (1.3)$$

for all  $x \in \mathbb{R}^n$ . If moreover  $u \geq 0$  in  $\mathbb{R}^n$  and  $\int_{\mathbb{R}^n} f(x, u) dx > 0$ , then we have

$$u(x) \geq C_1 (1 + |x|^{\frac{n-p}{p-1}})^{-1} \quad (1.4)$$

for all  $x \in \mathbb{R}^n$ , for some constant  $C_1 = C_1(n, p, \lambda, \Lambda, u) > 0$ , where  $\lambda$  is a real number such that  $0 < \lambda < \int_{\mathbb{R}^n} f(x, u) dx$ .

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The dependence on  $u$  of the constants  $C_0$  and  $C_1$  will be made more precise in Remarks 4.1 and 4.3.

In the case of the Laplace operator ( $p = 2$ ), the upper bound estimates (1.3) have been established by Jannelli–Solimini [14] for nonlinearities of the form  $f(x, u) = \sum_{i=1}^N a_i(x) |u|^{q_i^*-2} u$ , where  $q_i^* := 2^*(1 - 1/q_i)$ ,  $q_i \in (n/2, \infty]$ ,  $|a_i(x)| = O(|x|^{-n/q_i})$  for large  $|x|$ , and  $a_i$  belongs to the Marcinkiewicz space  $M^{q_i}(\mathbb{R}^n)$  for all  $i = 1, \dots, N$ . The case of unbounded domains  $\Omega \neq \mathbb{R}^n$  is also treated in [14].

Since the pioneer work of Gidas–Ni–Nirenberg [11] and later extensions by Li [18] in case  $p = 2$  and Damascelli–Ramaswamy [5] in case  $1 < p < 2$ , decay estimates are known to be useful to derive radial symmetry results for  $C^1$ -solutions of problems of the type

$$\begin{cases} -\Delta_p u = f(u), & u > 0 \quad \text{in } \mathbb{R}^n, \\ u(x) \longrightarrow 0 & \text{as } |x| \longrightarrow 0. \end{cases} \quad (1.5)$$

Here, we consider the following result of Damascelli–Ramaswamy [5] and Li [18]: if  $1 < p \leq 2$ ,  $f$  is a locally Lipschitz continuous function in  $(0, \infty)$  such that

$$\frac{f(v) - f(u)}{v - u} \leq \Lambda \max(u^\alpha, v^\alpha) \quad \forall u, v \text{ such that } 0 < u < v < s_0 \quad (1.6)$$

for some real numbers  $\Lambda, s_0 > 0$ , and  $\alpha > p - 2$ , and  $u$  is a  $C^1$ -solution of (1.5) such that

$$u(x) = O(|x|^{-m}) \quad \text{and} \quad |\nabla u(x)| = O(|x|^{-m-1}) \quad (1.7)$$

$$(\text{and } u(x) \geq C|x|^{-m} \text{ for large } |x| \text{ when } \alpha < 0) \quad (1.8)$$

for some real numbers  $C > 0$  and  $m > p/(\alpha + 2 - p)$ , then  $u$  is radially symmetric and strictly radially decreasing about some point  $x_0 \in \mathbb{R}^n$ , i.e. there exists  $v \in C^1(0, \infty)$  such that  $v'(r) < 0$  for all  $r > 0$  and  $u(x) = v(|x - x_0|)$  for all  $x \in \mathbb{R}^n$ . We also mention that other symmetry results for problems of type (1.5) have been established without any decay assumption in the case where  $f$  is nonincreasing near 0 (see Gidas–Ni–Nirenberg [11], Li [18], and Li–Ni [19] in case  $p = 2$ , Damascelli–Pacella–Ramaswamy [4], Damascelli–Ramaswamy [5], and Serrin–Zou [25] in case  $p \neq 2$ ).

In case  $\alpha = p^* - 2$ , the conditions (1.7)–(1.8) follow from (1.3)–(1.4) with  $m = (n - p)/(p - 1)$  (which is greater than  $p/(\alpha + 2 - p) = (n - p)/p$ ). Consequently, by combining Theorem 1.1, the results of Damascelli–Ramaswamy [5] and Li [18], and the regularity results that are referred to in Lemma 2.1 below, we obtain the following corollary.

**Corollary 1.2.** *Assume that  $1 < p \leq 2$ . Let  $f$  be a locally Lipschitz continuous function in  $(0, \infty)$  such that (1.2) and (1.6) hold true with  $f(x, s) = f(s)$ . Then any nonnegative solution of (1.1) is radially symmetric and strictly radially decreasing about some point  $x_0 \in \mathbb{R}^n$ .*

Let us now comment on the positive solutions of the equation with pure power nonlinearity, namely

$$-\Delta_p u = u^{p^*-1}, \quad u > 0 \quad \text{in } \mathbb{R}^n. \quad (1.9)$$

Guedda-Véron [13] proved that the only positive, radially symmetric solutions of (1.9) are of the form

$$u_{\mu, x_0}(x) = (n\mu)^{\frac{n-p}{p^2}} \left( \frac{n-p}{p-1} \right)^{\frac{(n-p)(p-1)}{p^2}} \left( \mu + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{p-n}{p}} \quad (1.10)$$

for all  $x \in \mathbb{R}^n$ , for some real number  $\mu > 0$  and point  $x_0 \in \mathbb{R}^n$ . In case  $p = 2$ , Caffarelli-Gidas-Spruck [2] (see also Chen-Li [3]) proved that the functions (1.10) are the only positive solutions of (1.9). In a recent paper, Damascelli-Merchán-Montoro-Sciunzi [6] proved that any solution in  $D^{1,p}(\mathbb{R}^n)$  of (1.9) is radially symmetric provided that  $2n/(n+2) \leq p < 2$ . The condition  $p \geq 2n/(n+2)$  corresponds to the values of  $p$  for which the function  $s \mapsto s^{p^*-1}$  is Lipschitz continuous near 0. With the above Corollary 1.2, we extend the result of Damascelli-Merchán-Montoro-Sciunzi [6] to the whole interval  $1 < p < 2$ . By combining the result of Guedda-Véron [13] and Corollary 1.2, we obtain the following corollary.

**Corollary 1.3.** *Assume that  $1 < p < 2$ . Then the functions (1.10) are the only positive solutions in  $D^{1,p}(\mathbb{R}^n)$  of (1.9).*

As a motivation to our results, it is well known that the profile of solutions of the equation

$$-\Delta_p u = |u|^{p^*-2} u \quad \text{in } \mathbb{R}^n \quad (1.11)$$

plays a central role in the blow-up theories of critical equations. Possible references in book form on this subject and its applications in case  $p = 2$  are Druet-Hebey-Robert [8], Ghoussoub [10], and Struwe [27]. In case  $p \neq 2$ , global compactness results in energy spaces in the spirit of Struwe [26] have been established in different contexts by Alves [1] for equations posed in the whole  $\mathbb{R}^n$ , Saintier [23] in the case of a smooth, compact manifold, and Mercuri-Willem [20] and Yan [32] in the case of a smooth, bounded domain. In view of these results, it is likely that the new information provided by Theorem 1.1 and Corollary 1.2 on the solutions of (1.11) will lead to new existence and multiplicity results as it is the case for  $p = 2$ .

The paper is organized as follows. Section 2 is concerned with global boundedness results. The key result in this section is a global bound in weak Lebesgue spaces which we obtain by arguments of measure theory. In Section 3, we establish a preliminary decay estimate which is not sharp but which turns out to be a crucial ingredient in what follows. To prove this estimate, we exploit the rescaling law of the equation, and

we apply a doubling property of Poláčik–Quittner–Souplet [22]. In Section 4, we conclude the proof of Theorem 1.1. The proof of the upper bound estimates (1.3) follows from the results of Sections 2 and 3 together with Harnack-type inequalities of Serrin [24] and Trudinger [30]. The proof of the lower bound estimate (1.4) requires a Harnack inequality on annuli in the spirit of Friedman–Véron [9].

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## 2. GLOBAL BOUNDEDNESS RESULTS

The first result of this section refers to some known regularity results for critical equations.

**Lemma 2.1.** *Let  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that (1.2) holds true. Then any solution of (1.1) belongs to  $W^{1,\infty}(\mathbb{R}^n) \cap C_{\text{loc}}^{1,\theta}(\mathbb{R}^n)$  for some  $\theta \in (0, 1)$ .*

*Proof of Lemma 2.1.* A straightforward adaptation of Peral [21, Theorem E.0.20] (which in turn is adapted from Trudinger [30, Theorem 3]) yields that for any solution  $u$  of (1.1), there exist constants  $C, R > 0$  and  $\beta > 1$  such that  $\|u\|_{L^{\beta p^*}(B(x,R))} \leq C$  for all  $x \in \mathbb{R}^n$ , where  $B(x, R)$  is the Euclidean ball of center  $x$  and radius  $R$ . We then obtain a global  $L^\infty$ -bound by applying Serrin [24, Theorem 1].

Once we have the  $L^\infty$ -boundedness of the solutions, the results of DiBenedetto [7] and Tolksdorf [28] provide global  $L^\infty$ -bounds and local Hölder regularity for the derivatives.  $\square$

The next result is concerned with the boundedness of solutions of (1.1) in weak Lebesgue spaces. For any  $s \in (0, \infty)$  and any domain  $\Omega \subset \mathbb{R}^n$ , we define  $L^{s,\infty}(\Omega)$  as the set of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\|u\|_{L^{s,\infty}(\Omega)} := \sup_{h>0} (h \cdot \text{meas}(\{|u| > h\})^{1/s}) < \infty,$$

where  $\text{meas}(\{|u| > h\})$  is the measure of the set  $\{x \in \Omega : |u(x)| > h\}$ . The map  $\|\cdot\|_{L^{s,\infty}(\Omega)}$  defines a quasi-norm on  $L^{s,\infty}(\Omega)$  (see for instance Grafakos [12]).

Our result is as follows.

**Lemma 2.2.** *Let  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that (1.2) holds true. Then any solution of (1.1) belongs to  $L^{p_*-1,\infty}(\mathbb{R}^n)$ , where  $p_* := p(n-1)/(n-p)$ , and hence by interpolation, any solution of (1.1) belongs to  $L^s(\mathbb{R}^n)$  for all  $s \in (p_* - 1, \infty]$ .*

*Proof of Lemma 2.2.* We let  $u$  be a nontrivial solution of (1.1). For any  $h > 0$ , by testing (1.1) with  $T_h(u) := \text{sgn}(u) \cdot \min(|u|, h)$ , where

$\text{sgn}(u)$  denotes the sign of  $u$ , we obtain

$$\int_{|u| \leq h} |\nabla u|^p dx = \int_{|u| \leq h} f(x, u) \cdot u dx + h \int_{|u| > h} f(x, u) \cdot \text{sgn}(u) dx. \quad (2.1)$$

It follows from (1.2) and (2.1) that

$$\int_{|u| \leq h} |\nabla u|^p dx \leq \Lambda \left( \int_{|u| \leq h} |u|^{p^*} dx + h \int_{|u| > h} |u|^{p^*-1} dx \right). \quad (2.2)$$

We then write

$$\int_{|u| \leq h} |u|^{p^*} dx = \int_{\mathbb{R}^n} |T_h(u)|^{p^*} dx - h^{p^*} \text{meas}(\{|u| > h\}) \quad (2.3)$$

and

$$\begin{aligned} \int_{|u| > h} |u|^{p^*-1} dx &= (p^* - 1) \int_0^\infty s^{p^*-2} \text{meas}(\{|u| > \max(s, h)\}) ds \\ &= h^{p^*-1} \text{meas}(\{|u| > h\}) + (p^* - 1) \int_h^\infty s^{p^*-2} \text{meas}(\{|u| > s\}) ds. \end{aligned} \quad (2.4)$$

It follows from (2.2)–(2.4) that

$$\begin{aligned} \int_{|u| \leq h} |\nabla u|^p dx &\leq \Lambda \left( \int_{\mathbb{R}^n} |T_h(u)|^{p^*} dx \right. \\ &\quad \left. + (p^* - 1) h \int_h^\infty s^{p^*-2} \text{meas}(\{|u| > s\}) ds \right). \end{aligned} \quad (2.5)$$

Sobolev inequality gives

$$\int_{\mathbb{R}^n} |T_h(u)|^{p^*} dx \leq K \left( \int_{|u| \leq h} |\nabla u|^p dx \right)^{\frac{n}{n-p}} \quad (2.6)$$

for some constant  $K = K(n, p)$ . By (2.3), (2.5), (2.6), and since  $\int_{\mathbb{R}^n} |T_h(u)|^{p^*} dx = o(1)$  as  $h \rightarrow 0$ , we obtain

$$\begin{aligned} h^{p^*} \text{meas}(\{|u| > h\}) &\leq \int_{\mathbb{R}^n} |T_h(u)|^{p^*} dx \\ &\leq C \left( h \int_h^\infty s^{p^*-2} \text{meas}(\{|u| > s\}) ds \right)^{\frac{n}{n-p}} \end{aligned} \quad (2.7)$$

for small  $h$ , for some constant  $C = C(n, p, \Lambda)$ . We then define

$$F(h) := \left( \int_h^\infty f(s) ds \right)^{\frac{-p}{n-p}}, \quad \text{where } f(s) := s^{p^*-2} \text{meas}(\{|u| > s\}).$$

Since the function  $t \mapsto t^{-p/(n-p)}$  is locally Lipschitz in  $(0, \infty)$  and  $\int_h^\infty f(s) ds > 0$  for all  $h < \|u\|_{L^\infty(\mathbb{R}^n)}$ , we get that  $F$  is locally absolutely continuous in  $(0, \|u\|_{L^\infty(\mathbb{R}^n)})$  with derivative

$$F'(h) = \frac{p}{n-p} \left( \int_h^\infty f(s) ds \right)^{\frac{-n}{n-p}} f(h) \quad (2.8)$$

for a.e.  $h \in (0, \|u\|_{L^\infty(\mathbb{R}^n)})$  (see for instance Leoni [17, Theorem 3.68]). By (2.7) and (2.8), we obtain

$$F'(h) \leq C \cdot \frac{p}{n-p} \cdot h^{\frac{2p-n}{n-p}} \quad (2.9)$$

for small  $h$ . Integrating (2.9) gives

$$F(h) - F(0) \leq Ch^{\frac{p}{n-p}} \quad (2.10)$$

for small  $h$ , where  $F(0) := \lim_{h \rightarrow 0} F(h)$ . On the other hand, by (2.4) and dominated convergence, we have

$$(p^* - 1)hF(h)^{\frac{p-n}{p}} \leq h \int_{|u|>h} |u|^{p^*-1} dx = o(1) \quad (2.11)$$

as  $h \rightarrow 0$ . It follows from (2.10) and (2.11) that  $F(0) > 0$ , i.e.  $\int_0^\infty f(s) ds < \infty$ . By (2.7) and since  $p^* - \frac{n}{n-p} = p_* - 1$  and  $F$  is nonincreasing, we then get

$$h^{p_*-1} \text{meas}(\{|u| > h\}) \leq CF(h)^{-n/p} \leq CF(0)^{-n/p}$$

for small  $h$ , and hence we obtain  $\|u\|_{L^{p_*-1, \infty}(\mathbb{R}^n)} < \infty$ .  $\square$

By (1.2) and a weak version of Kato's inequality [15] (see Leon [16, Proposition 3.2]), we obtain

$$-\Delta_p |u| \leq |f(x, u)| \leq \Lambda |u|^{p^*-1} \quad \text{in } \mathbb{R}^n, \quad (2.12)$$

where the inequality is in the sense that

$$\int_{\mathbb{R}^n} |\nabla |u||^{p-2} \nabla |u| \cdot \nabla \varphi dx \leq \Lambda \int_{\mathbb{R}^n} |u|^{p^*-1} \varphi dx$$

for all nonnegative, smooth function  $\varphi$  with compact support in  $\mathbb{R}^n$ .

Our last result in this section is as follows.

**Lemma 2.3.** *For any real number  $\Lambda > 0$  and any nonnegative, non-trivial solution  $v \in D^{1,p}(\mathbb{R}^n)$  of the inequality  $-\Delta_p v \leq \Lambda v^{p^*-1}$  in  $\mathbb{R}^n$ , we have  $\|v\|_{L^{p^*}(\mathbb{R}^n)} \geq \kappa_0$  for some constant  $\kappa_0 = \kappa_0(n, p, \Lambda) > 0$ .*

*Proof.* By Sobolev inequality and testing the inequality  $-\Delta_p v \leq \Lambda v^{p^*-1}$  with the function  $v$ , we obtain

$$\int_{\mathbb{R}^n} v^{p^*} dx \leq K \left( \int_{\mathbb{R}^n} |\nabla v|^p dx \right)^{\frac{n}{n-p}} \leq K \left( \Lambda \int_{\mathbb{R}^n} v^{p^*} dx \right)^{\frac{n}{n-p}} \quad (2.13)$$

for some constant  $K = K(n, p)$ . The result then follows immediately from (2.13).  $\square$

## 3. A PRELIMINARY DECAY ESTIMATE

The following result provides a decay estimate which is not sharp but which will serve as a preliminary step in the proof of Theorem 1.1. The proof relies on rescaling arguments and a doubling property of Poláčik–Quittner–Souplet [22].

**Lemma 3.1.** *Let  $\kappa_0$  be as in Lemma 2.3,  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that (1.2) holds true, and  $u$  be a solution of (1.1). For any  $\kappa > 0$ , we define*

$$r_\kappa(u) := \inf \left( \{r > 0 : \|u\|_{L^{p^*}(\mathbb{R}^n \setminus B(0,r))} < \kappa\} \right),$$

where  $B(0, r)$  denotes the Euclidean ball of center 0 and radius  $r$ . Then for any  $\kappa \in (0, \kappa_0)$  and  $r > r_\kappa(u)$ , there exists a constant  $K_0 = K_0(n, p, \Lambda, \kappa, r, r_\kappa(u))$  such that

$$|u(x)| \leq K_0 |x|^{\frac{p-n}{p}} \quad \text{for all } x \in \mathbb{R}^n \setminus B(0, r). \quad (3.1)$$

*Proof of Lemma 3.1.* We fix  $\Lambda > 0$ ,  $\kappa \in (0, \kappa_0)$ ,  $r > 0$ , and  $r' \in (0, r)$ . As is easily seen, in order to prove Lemma 3.1, it is sufficient to prove that there exists a constant  $K_1 = K_1(n, p, \Lambda, \kappa, r, r')$  such that for any solution  $u$  of (1.1) such that  $r_\kappa(u) \leq r'$ , we have

$$\text{dist}(x, B(0, r'')) |u(x)|^{\frac{p}{n-p}} \leq K_1 \quad \text{for all } x \in \mathbb{R}^n \setminus B(0, r), \quad (3.2)$$

where  $r'' := (r + r')/2$  and  $\text{dist}$  is the Euclidean distance function.

We prove (3.2) by contradiction. Suppose that for any  $\alpha \in \mathbb{N}$ , there exists a Caratheodory function  $f_\alpha : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  such that (1.2) holds true, a solution  $u_\alpha$  of (1.1) with  $f = f_\alpha$  such that  $r_\kappa(u_\alpha) \leq r'$ , and a point  $x_\alpha \in \mathbb{R}^n \setminus B(0, r)$  such that

$$\text{dist}(x_\alpha, B(0, r'')) |u_\alpha(x_\alpha)|^{\frac{p}{n-p}} > 2\alpha. \quad (3.3)$$

By (3.3) and Poláčik–Quittner–Souplet [22, Lemma 5.1], we get that there exists  $y_\alpha \in \mathbb{R}^n \setminus B(0, r'')$  such that

$$\text{dist}(y_\alpha, B(0, r'')) |u_\alpha(y_\alpha)|^{\frac{p}{n-p}} > 2\alpha, \quad |u_\alpha(x_\alpha)| \leq |u_\alpha(y_\alpha)|, \quad (3.4)$$

and

$$|u_\alpha(y)| \leq 2^{\frac{n-p}{p}} |u_\alpha(y_\alpha)| \quad \text{for all } y \in B(y_\alpha, \alpha |u_\alpha(y_\alpha)|^{\frac{-p}{n-p}}). \quad (3.5)$$

For any  $\alpha$  and  $y \in \mathbb{R}^n$ , we define

$$\tilde{u}_\alpha(y) := \mu_\alpha \cdot u_\alpha(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha), \quad (3.6)$$

where  $\mu_\alpha := |u_\alpha(y_\alpha)|^{-1}$ . By rescaling (1.1), we obtain

$$-\Delta_p \tilde{u}_\alpha = \mu_\alpha^{p^*-1} \cdot f_\alpha(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha, \mu_\alpha^{-1} \cdot \tilde{u}_\alpha) \quad \text{in } \mathbb{R}^n. \quad (3.7)$$

It follows from (1.2) that

$$|\mu_\alpha^{p^*-1} \cdot f_\alpha(\mu_\alpha^{\frac{p}{n-p}} \cdot y + y_\alpha, \mu_\alpha^{-1} \cdot \tilde{u}_\alpha)| \leq \Lambda |\tilde{u}_\alpha|^{p^*-1} \quad \text{in } \mathbb{R}^n. \quad (3.8)$$

Moreover, by (3.5) and (3.6), we obtain

$$|\tilde{u}_\alpha(0)| = 1 \quad \text{and} \quad |\tilde{u}_\alpha(y)| \leq 2^{\frac{n-p}{p}} \quad \text{for all } y \in B(0, \alpha). \quad (3.9)$$

By DiBenedetto [7] and Tolksdorf [28], it follows from (3.8) and (3.9) that there exists a constant  $C > 0$  and a real number  $\theta \in (0, 1)$  such that for point  $x \in \mathbb{R}^n$ , we have

$$\|\tilde{u}_\alpha\|_{C^{1,\theta}(B(x,1))} \leq C \quad (3.10)$$

for large  $\alpha$ . By compactness of  $C^{1,\theta}(B(x,1)) \hookrightarrow C^1(B(x,1))$ , it follows from (3.10) that  $(\tilde{u}_\alpha)_\alpha$  converges up to a subsequence in  $C^1_{\text{loc}}(\mathbb{R}^n)$  to some function  $\tilde{u}_\infty$ . By (3.9), we obtain  $|\tilde{u}_\infty(0)| = 1$ . Moreover, by observing that both the  $D^{1,p}$ -norm and the inequality (2.12) are left invariant by the rescaling (3.7), we get that  $\tilde{u}_\infty \in D^{1,p}(\mathbb{R}^n)$  and  $|\tilde{u}_\infty|$  is a weak solution of

$$-\Delta_p |\tilde{u}_\infty| \leq \Lambda |\tilde{u}_\infty|^{p^*-1} \quad \text{in } \mathbb{R}^n. \quad (3.11)$$

On the other hand, for any  $R > 0$ , we have

$$\|\tilde{u}_\alpha\|_{L^{p^*}(B(0,R))} = \|u_\alpha\|_{L^{p^*}(B(y_\alpha, R\mu_\alpha^{\frac{p}{n-p}}))}. \quad (3.12)$$

By (3.4) and since  $r_\kappa(u_\alpha) < r''$ , we get

$$B(y_\alpha, R\mu_\alpha^{\frac{p}{n-p}}) \cap B(0, r_\kappa(u_\alpha)) = \emptyset \quad (3.13)$$

for large  $\alpha$ . By (3.12), (3.13), and by definition of  $r_\kappa(u_\alpha)$ , we obtain

$$\|\tilde{u}_\alpha\|_{L^{p^*}(B(0,R))} \leq \kappa \quad (3.14)$$

for large  $\alpha$ . Passing to the limit into (3.14) as  $\alpha \rightarrow \infty$  and then as  $R \rightarrow \infty$  yields

$$\|\tilde{u}_\infty\|_{L^{p^*}(\mathbb{R}^n)} \leq \kappa. \quad (3.15)$$

Since  $\kappa < \kappa_0$ , by Lemma 2.3, (3.11), and (3.15), we get that  $\tilde{u}_\infty \equiv 0$ , which is in contradiction with  $|\tilde{u}_\infty(0)| = 1$ . This ends the proof of Lemma 3.1.  $\square$

#### 4. PROOF OF THEOREM 1.1

We can now prove Theorem 1.1 by applying Lemmas 2.2, 3.1, and Harnack-type inequalities of Serrin [24] and Trudinger [30].

*Proof of (1.3).* We let  $u$  be a solution of (1.1). We let  $\kappa$  and  $r$  be as in Lemma 3.1. For any  $R > 0$  and  $y \in \mathbb{R}^n$ , we define

$$u_R(y) := R^{\frac{n-p}{p-1}} \cdot u(R \cdot y). \quad (4.1)$$

By rescaling (1.1), we obtain

$$-\Delta_p u_R = R^n \cdot f(R \cdot y, R^{\frac{p-n}{p-1}} \cdot u_R) \quad \text{in } \mathbb{R}^n. \quad (4.2)$$

It follows from (1.2) that

$$|R^n \cdot f(R \cdot y, R^{\frac{p-n}{p-1}} \cdot u_R)| \leq \Lambda \cdot R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \quad \text{in } \mathbb{R}^n. \quad (4.3)$$



Moreover, similarly to (2.12), it follows from (4.2) and (4.3) that  $|u_R|$  is a weak solution of

$$-\Delta_p |u_R| \leq \Lambda \cdot R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \quad \text{in } \mathbb{R}^n. \quad (4.4)$$

By writing  $|u_R|^{p^*-1} = |u_R|^{p^*-p} \cdot |u_R|^{p-1}$  and applying Lemma 3.1, we obtain

$$R^{\frac{-p}{p-1}} \cdot |u_R|^{p^*-1} \leq K_0^{p^*-p} |u_R|^{p-1} \quad \text{in } \mathbb{R}^n \setminus B(0, 1) \quad (4.5)$$

provided that  $R \geq r$ . It follows from (4.4), (4.5), and Trudinger [30, Theorem 1.3] that for any  $\varepsilon > 0$ , we have

$$\|u_R\|_{L^\infty(B(0,2) \setminus B(0,4))} \leq c_\varepsilon \|u_R\|_{L^{p-1+\varepsilon}(B(0,5) \setminus B(0,1))}. \quad (4.6)$$

for some constant  $c_\varepsilon = c(n, p, \Lambda, K_0, \varepsilon)$ . We fix  $\varepsilon_0 = \varepsilon_0(n, p)$  such that  $0 < \varepsilon_0 < p_* - p$ , where  $p_*$  is as in Lemma 2.2. By a generalized version of Hölder's inequality (see for instance Grafakos [12, Exercise 1.1.11]), we obtain that there exists a constant  $c_0 = c_0(n, p)$  such that

$$\|u_R\|_{L^{p-1+\varepsilon_0}(B(0,5) \setminus B(0,1))} \leq c_0 \|u_R\|_{L^{p_*-1,\infty}(B(0,5) \setminus B(0,1))}. \quad (4.7)$$

By observing that the quasi-norm  $\|\cdot\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}$  is left invariant by the rescaling (4.1), we deduce from (4.6), (4.7), and Lemma 2.2 that

$$\|u_R\|_{L^\infty(B(0,2) \setminus B(0,4))} \leq c_1 \quad (4.8)$$

for some constant  $c_1 = c_1(n, p, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)})$ . By (4.2)–(4.5), (4.8), and the estimates of DiBenedetto [7] and Tolksdorf [28], we get

$$\|\nabla u_R\|_{L^\infty(B(0,5/2) \setminus B(0,7/2))} \leq c_2. \quad (4.9)$$

for some constant  $c_2 = c_2(n, p, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)})$ . Finally, for any  $x \in \mathbb{R}^n \setminus B(0, 3r)$ , by applying (4.8) and (4.9) with  $R = |x|/3$ , we obtain

$$|u(x)| \leq c_3 |x|^{\frac{p-n}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq c_3 |x|^{\frac{1-n}{p-1}} \quad (4.10)$$

for some constant  $c_3 = c_3(n, p, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)})$ . Since on the other hand  $u$  and  $\nabla u$  are uniformly bounded in  $B(0, 3r)$ , we can deduce (1.3) from (4.10).  $\square$

**Remark 4.1.** *As one can see from the above proof, the constant  $C_0$  in (1.3) depends only on  $n, p, \Lambda, \kappa, r$ , and upper bounds of  $r_\kappa(u)$ ,  $\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}$ , and  $\|u\|_{W^{1,\infty}(B(0,3r))}$ .*

In order to prove the lower bound estimate (1.4), we need the following Harnack inequality on annuli. This result is inspired from Friedman–Véron [9] where a similar result is used for the study of singular solutions of  $p$ -Laplace equations in pointed domains.

**Lemma 4.2.** *Let  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that (1.2) holds true,  $u$  be a nonnegative solution of (1.1),  $\kappa$  and  $r$  be as in Lemma 3.1, and  $K_0$  be the constant given by Lemma 3.1. Then there exists a constant  $c_4 = c_4(n, p, \Lambda, K_0)$  such that*

$$\sup_{2R < |x| < 5R} (u(x)) \leq c_4 \cdot \inf_{2R < |x| < 5R} (u(x)) \quad (4.11)$$

for all  $R \geq r$ .

*Proof of Lemma 4.2.* For any  $R > 0$ , we define  $u_R$  as in (4.1). By (4.2), (4.3), (4.5), and Serrin [24, Theorem 5], we obtain that there exists a constant  $c = c(n, p, \Lambda, K_0)$  such that

$$\sup_{z \in B(y, 1/3)} (u_R(z)) \leq c \cdot \inf_{z \in B(y, 1/3)} (u_R(z)) \quad (4.12)$$

for all points  $y$  in the annulus  $A := B(0, 5) \setminus B(0, 2)$ . Moreover, we can join every two points in  $A$  by 17 connected balls of radius  $1/3$  and centers in  $A$ . Hence (4.11) follows from (4.12) with  $c_4 := c^{17}$ .  $\square$

We can now prove (1.4) by applying Lemma 4.2.

*Proof of (1.4).* We let  $u$  be a nonnegative solution of (1.1) such that  $\int_{\mathbb{R}^n} f(x, u) dx > 0$ . In particular, in view of (1.2), we have  $u \not\equiv 0$ , and hence  $u > 0$  in  $\mathbb{R}^n$  by the strong maximum principle of Vazquez [31]. By Lemma 4.2, we then get that in order to prove (1.4), it is sufficient to prove a lower bound estimate of  $\|u\|_{L^\infty(B(0, 5R) \setminus B(0, 2R))}$  for large  $R$ .

By (4.2), (4.3), (4.5), and Serrin [24, Theorem 1], we obtain

$$\begin{aligned} \|\nabla u_R\|_{L^p(B(0, 4) \setminus B(0, 3))} &\leq c_5 \|u_R\|_{L^p(B(0, 5) \setminus B(0, 2))} \\ &\leq c'_5 \|u_R\|_{L^\infty(B(0, 5) \setminus B(0, 2))} \end{aligned} \quad (4.13)$$

for some constants  $c_5$  and  $c'_5$  depending only on  $n, p, \Lambda$ , and  $K_0$ , where  $u_R$  is as in (4.1). Rescaling (4.13) yields

$$\|\nabla u\|_{L^p(B(0, 4R) \setminus B(0, 3R))} \leq c'_5 R^{\frac{n-p}{p}} \|u\|_{L^\infty(B(0, 5R) \setminus B(0, 2R))}. \quad (4.14)$$

Next, we claim that if  $\int_{\mathbb{R}^n} f(x, u) dx > \lambda$  for some real number  $\lambda > 0$ , then we have

$$\|\nabla u\|_{L^p(B(0, 4R) \setminus B(0, 3R))} \geq c_6 R^{\frac{p-n}{p(p-1)}} \quad (4.15)$$

for large  $R$ , for some constant  $c_6 = c_6(n, p, \lambda) > 0$ . For any  $x \in \mathbb{R}^n$  and  $R > 0$ , we define  $\chi_R(x) := \chi(|x|/R)$ , where  $\chi \in C^1(0, \infty)$  is a cutoff function such that  $\chi \equiv 1$  on  $[0, 3]$ ,  $\chi \equiv 0$  on  $[4, \infty)$ ,  $0 \leq \chi \leq 1$  and  $|\chi'| \leq 2$  on  $(3, 4)$ . By testing (1.1) with  $\chi_R$  and applying Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f(x, u) \chi_R dx &= \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \chi_R dx \\ &\leq \|\nabla u\|_{L^p(\text{supp}(\nabla \chi_R))}^{p-1} \cdot \|\nabla \chi_R\|_{L^p(\text{supp}(\nabla \chi_R))}, \end{aligned} \quad (4.16)$$

where  $\text{supp}(\chi_R)$  denotes the support of  $\chi_R$ . It follows from (4.16) and the definition of  $\chi_R$  that

$$\int_{\mathbb{R}^n} f(x, u) \chi_R dx \leq C R^{\frac{n-p}{p}} \|\nabla u\|_{L^p(B(0,4R) \setminus B(0,3R))}^{p-1} \quad (4.17)$$

for some constant  $C = C(n, p) > 0$ . Then (4.15) follows from (4.17) with  $c_6 := (\lambda/C)^{\frac{1}{p-1}}$

Finally, we deduce (1.4) from (4.11), (4.14), and (4.15).  $\square$

**Remark 4.3.** *As one can see from the above proof, the constant  $C_1$  in (1.4) depends only on  $n, p, \lambda, \Lambda, \kappa, r, r_\kappa(u)$ , and a lower bound of  $u$  on the ball  $B(0, 2 \max(r, R_{\lambda,f}(u)))$ , where*

$$R_{\lambda,f}(u) := \inf \left( \left\{ R > 0 : \int_{\mathbb{R}^n} f(x, u) \chi_{R'} dx > \lambda, \quad \forall R' > R \right\} \right).$$

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